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Scalar Field Cosmology I: Asymptotic Freedom and the Initial-Value Problem

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Abstract

The purpose of this work is to use a renormalized quantum scalar field to investigate very early cosmology, in the Planck era immediately following the big bang. Renormalization effects make the field potential dependent on length scale, and are important during the big bang era. We use the asymptotically free Halpern-Huang scalar field, which is derived from renormalization-group analysis, and solve Einstein's equation with Robertson-Walker metric as an initial-value problem. The main prediction is that the Hubble parameter follows a power law: $H \equiv \dot{a}/a \sim t^{-p}$, and the universe expands at an accelerated rate: $a \sim \exp t^{1-p}$. This gives "dark energy", with an equivalent cosmological constant that decays in time like t^{-2p}, which avoids the "fine-tuning" problem. The power law predicts a simple relation for the galactic redshift. Comparison with data leads to the speculation that the universe experienced a crossover transition, which was completed about 7 billion years ago.

CONTENTS

I. INTRODUCTION AND SUMMARY

According to quantum field theory, the vacuum is not empty and static, but filled with fluctuating quantum fields. Those of the electromagnetic field, which fluctuate about zero, can be measured experimentally through the Lamb shift in the hydrogen spectrum, and the electron's anomalous magnetic moment. Others, such as the scalar Higgs field of the standard model, fluctuate about a nonzero vacuum field. Grand unified models call for still more vacuum scalar fields. These vacuum scalar fields are similar to the Ginsburg-Landau order parameter in superconductivity, which is a phenomenological way to describe the condensate of Cooper pairs of the more fundamental BCS theory. Be they elementary or phenomenological, these vacuum fields behave like classical fields in many respects. Under certain conditions, however, one must take into account their quantum nature. In particular, during the big bang, when the length scale of the universe undergoes rapid change, one

must take into account the effects of renormalization, and this is the focus of the present investigation. Some of our results have been reported in a previous note [1].

Scalar fields have been used in traditional cosmological theories to explain "dark energy" [2], and "cosmic inflation" [3]. Dark energy refers to an accelerating expansion of the universe, which can be reproduced by introducing a "cosmological constant" in Einstein's equation. This is equivalent to introducing a static scalar field with constant energy density. The problem is that the cosmological constant is naturally measured on the Planck scale, which is some 60 orders of magnitude greater than that fitted to presently observed data. One would have to "fine-tune" it (by 60 orders of magnitude!), and this has been deemed unpalatable.

The theory of cosmic inflation, designed to explain the presently observed large-scale uniformity of the universe, postulates that matter was created while the universe was so small that all matter "saw" each other. The universe then expanded by an enormous order of magnitude (e.g., 27) in an extremely short time (e.g., 10^{-26}s), pushing part of the matter beyond the event horizon of other parts, but the original density was retained. To implement this scenario, one introduces a scalar field with spontaneous symmetry breaking, i.e., having a potential with a minimum located at a nonzero value of the field. Initially the universe was placed at the "false vacuum" of zero field, and it is supposed to inflate during the time it takes to "roll down" the potential towards the true vacuum. It would be desirable to formulate this scenario in terms of a mathematically consistent initial-value problem. However, this has not been done so far. As we shall see, the universe does inflate in our model, but the "rolling" was anything but slow.

Most previous works on vacuum scalar fields treat them classically, i.e., with fixed given potentials. In quantum field theory, however, the potential is subject to renormalization, and changes with the energy scale. This arises from the fact that there exist virtual processes with momenta extending all the way to infinity. The high end of the spectrum causes divergences in the theory, and in any case does not correspond to the true physics. To make the theory mathematically defined, the spectrum must be cut off at some momentum Λ, and this cutoff is the only scale parameter in a self-contained field theory. When Λ changes, all coupling constants must change in such a manner as to preserve the theory (i.e., to preserve all the correlation functions), and this change is called "renormalization". Such cutoff dependence can be ignored when one studies phenomena at a fixed length scale, such as stellar structure

at a particular epoch of the universe; but it is all-important during the big bang.

The purpose of this work is to study the implications of renormalized quantum scalar fields in the immediate neighborhood of the big bang. The mathematical problem is to formulate and solve an initial-value problem based on Einstein's equation, with suitable idealizations to render the problem tractable. This basic principle is that there is only one scale in the early cosmos, namely the "radius" a of the universe set by the metric tensor. Thus, we must identify a with inverse cutoff momentum Λ^{-1}. For consistency, the self-interaction potential of the scalar field should be "asymptotically free", i.e., vanish in the limit $a \to 0$.

From renormalization-group (RG) analysis, Halpern and Huang (HH) [4] have shown that asymptotic freedom determines the potential of the scalar field to be a Kummer function, a transcendental function that has exponential behavior for large fields, and this rules out all polynomial potentials, including the popular ϕ^4 theory. In the present work, we use the HH scalar field as the source of gravity, in Einstein's equation with Robertson-Walker (RW) metric. As mentioned earlier, our basic principle is that the inverse radius of the universe acts as the momentum cutoff of the scalar field theory, i.e., $\Lambda = a^{-1}$. This gives rise to a dynamical feedback: the expansion of the universe is driven by the scalar field, whose potential depends on the radius of the universe.

The main prediction of the model is that the Hubble parameter $H = \dot{a}/a$ behaves like a power $H \sim t^{-p}$ $(0 < p < 1)$, for large times, after averaging over small rapid oscillations. The exponent p depends on model parameters and initial conditions. This indicates "dark energy", for the universe expands with acceleration, according to $a \sim \exp t^{1-p}$. This behavior corresponds to an equivalent cosmological constant that decays with time like t^{-2p}, and this avoids the usual fine-tuning problem. The origin of the power law can be traced to a constraint on initial values from the 00 component of Einstein's equation.

Although our model is valid only in a neighborhood of the big bang, it is hard to resist to compare it with observations from a much later universe. A partial justification for doing this is that the power-law character may survive generalizations of the model. In this spirit, we calculate the relation between luminosity distance d_L and red shift z for a light source, according to the power law. To an extremely good approximation, we find $d_L(z) = z(1+z)\,d_0$, in which the exponent p enters only through the constant d_0. Comparison with data on the galactic redshift, from supernova and gamma-ray burst measurements,

suggest that there was an epoch in which d_0 had a different value from the current one, and connecting the two epochs was a crossover transition completed about 7 billion years ago.

Finally we address the scenario of cosmic inflation, which is inseparable with matter creation. The question is whether enough matter can be created for subsequent nucleosynthesis, during the time when the universe was small enough that all constituents remained within each other's event horizon.

An equally important question has to do with the emergence of the matter energy scale, which is from the Planck scale by some 18 orders of magnitude. Physically, the matter scale arises spontaneously, via "dimensional transmutation" in QCD, and in our model it enters through the coupling parameter between matter and the scalar field. These two scales must decouple from each other. How does it happen mathematically in our model?

To explore these questions, we treat matter a perfect fluid coupled to the scalar field, as detailed in Appendix C. Our studies lead to the opinion that a completely spatially homogeneous scalar field, real or complex, cannot give a satisfactory inflation scenario. First, it cannot created enough matter in a short enough time, and secondly decoupling does not occur, whatever one chooses for the matter coupling parameters. The model so far appears to lack physical mechanisms for matter creation and decoupling.

We are led to investigate a complex scalar field with uniform modulus but spatially varying phase. This makes the universe a superfluid, and new physics emerges, namely vorticity and quantum turbulence. We find that these phenomena can supply the missing mechanisms for matter creation and decoupling. This development is the subject of paper II of this series [5].

II. PRELIMINARIES

We start with Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1}$$

where $g^{\mu\nu}$ is the metric tensor that reduces to the diagonal form $(-1, 1, 1, 1)$ in flat spacetime, $T_{\mu\nu}$ is the energy-momentum tensor of non-gravitational fields, and $G = 6.672 \times 10^{-11}$ m^3 kg^{-1} s^{-2} is the gravitational constant. We shall put $4\pi G = 1$, thus measuring everything

in Planck units [6]:

$$\begin{aligned}\text{Planck length} &= \left(\hbar c^{-3}\right)^{1/2}(4\pi G)^{1/2} = 5.73\times 10^{-35}\text{ m}\\ \text{Planck time} &= \left(\hbar c^{-5}\right)^{1/2}(4\pi G)^{1/2} = 1.91\times 10^{-43}\text{ s}\\ \text{Planck energy} &= \left(\hbar c^{5}\right)^{1/2}(4\pi G)^{-1/2} = 3.44\times 10^{18}\text{ GeV}\end{aligned} \tag{2}$$

Consider a spatially homogeneous universe defined by the Robertson-Walker (RW) metric, which is specified through the line element

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{3}$$

where t is the time, $\{r,\theta,\phi\}$ are dimensionless spherical coordinates, and $a\left(t\right)$ is the length scale. The curvature parameter is $k = 0, \pm 1$, where $k = 1$ corresponds to a space with positive curvature, $k = -1$ that with negative curvature, and $k = 0$ is the limiting case of zero curvature. With the RW metric, the 00 and ij component of Einstein's equation reduce to the following Friedman equations:

$$\begin{aligned}\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} &= -\frac{2}{3}T_{00}\\ \left[\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2}\right]g_{ij} &= -2T_{ij} \qquad (i,j = 1,2,3)\end{aligned} \tag{4}$$

It is customary to introduce the Hubble parameter defined by

$$H = \frac{\dot a}{a} \tag{5}$$

The energy-momentum tensor of a spatially uniform system must have the form

$$\begin{aligned}T^{00} &= \rho\\ T^{ij} &= g^{ij}p\\ T^{j0} &= 0\end{aligned} \tag{6}$$

where ρ defines the energy density, and p the pressure. Energy-momentum conservation is expressed by $T^{\mu\nu}_{;\mu} = 0$, which, with the RW metric, becomes

$$\dot\rho + \frac{3\dot a}{a}\left(\rho + p\right) = 0 \tag{7}$$

We can recast the Friedman equations in terms of H, and, with inclusion of the conservation equation, obtain three cosmological equations:

$$
\begin{aligned}
\dot{H} &= \frac{k}{a^2} - (p + \rho) \\
H^2 &= -\frac{k}{a^2} + \frac{2}{3}\rho \\
\dot{\rho} &= -3H\,(\rho + p)
\end{aligned}
\tag{8}
$$

The second equation is a constraint of the form

$$
X \equiv H^2 + \frac{k}{a^2} - \frac{2}{3}\rho = 0 \tag{9}
$$

The third equation, the conservation law, states $\dot{X} = 0$, i.e., the constraint is a constant of the motion.

As an example, consider Einstein's cosmological constant Λ_0, which appears in a static energy-momentum tensor of the form (with units restored for convenience)

$$
T_{0\mu\nu} = -g_{\mu\nu}\,(\Lambda_0/8\pi G) \tag{10}
$$

Corresponding to this, the energy density and pressure are given by

$$
\begin{aligned}
\rho_0 &= \Lambda_0/8\pi G \\
p_0 &= -\Lambda_0/8\pi G
\end{aligned}
\tag{11}
$$

The conservation equation now states $\dot{\rho}_0 = 0$, which is trivial. Thus the cosmological equations reduce to

$$
\begin{aligned}
\dot{H} &= \frac{k}{a^2} \\
H^2 &= -\frac{k}{a^2} + \frac{2}{3}\rho_0
\end{aligned}
\tag{12}
$$

The asymptotic solution describes an exponentially expanding universe, with

$$
\begin{aligned}
a\,(t) &\sim \exp\,(H_\infty t) \\
H_\infty &= (\Lambda_0/12\pi G)^{1/2}
\end{aligned}
\tag{13}
$$

Since $a\,(t)$ is accelerating, we can say that there is "dark energy". However, the "natural" value of H_∞ should be of order unity on the Planck scale, whereas the presently observed

Hubble parameter is of order 10^{-60}. One would have to "fine tune" H_∞, by sixty orders of magnitude.

With a dynamical scalar field, the constraint implies $H_\infty = 0$. This is illustrated in Appendix A in an exact solution for the massless free scalar field, in which $\dot{a}/a$ decays according to a power law, which is equivalent to saying that H_∞ decays like a power. The effective cosmological constant is being "fine-tuned to zero", so to speak. This "automatic fine-tuning" also happens in our model, to be discussed later.

III. HALPERN-HUANG SCALAR FIELD

The HH scalar field that we use in this work has an asymptotically free potential, which is summarized here. Appendix B give a derivation from renormalization theory.

For generality, consider an N-component real scalar field $\phi_n(x)$ with $O(N)$ symmetry, with Lagrangian density (with $\hbar = c = 1$)

$$\mathcal{L}_{\mathrm{sc}}(x) = -\frac{1}{2} g^{\mu\nu} \sum_{n=1}^{N} \partial_\mu \phi_n \partial_\nu \phi_n - V(\phi) \tag{14}$$

where $\phi^2 = \sum_{n=1}^{N} \phi_n^2$. The high-energy cutoff Λ is introduced through a modification of the two-particle propagator at small distances. (See Appendix B for details.) The form of the modification is not important here; what is important is that Λ is the only intrinsic scale of the scalar field. All coupling constants g_n in the power-series $V = \sum_n g_n \phi^n$ must scale with appropriate powers of Λ. In 4-dimensional space-time we have $g_n = \Lambda^{4-n} u_n$, where the u_n are dimensionless, but depend on Λ; they undergo "renormalization" in order to preserve the theory. As Λ changes, $\{u_n\}$ trace out an RG trajectory in parameter space. There exist fixed points in this space, representing scale-invariant systems with $\Lambda = \infty$. A obvious fixed point is the Gaussian fixed point corresponding to $V \equiv 0$, i.e., the massless free field.

In a universe governed by the RW metric with length scale a, we must identify

$$\Lambda = \frac{\hbar}{a} \tag{15}$$

where we restore Planck's constant $\hbar$ to remind us of the quantum nature of the cutoff. The big bang corresponds to $a = 0$, or the Gaussian fixed point. In a consistent theory, therefore, the potential must vanish as $a \to 0$, or $\Lambda \to \infty$. In the language of particle physics, the theory must be "asymptotically free". We imagine that at the big bang, the scalar field was

displaced infinitesimally from the Gaussian fixed point onto some RG trajectory, along some direction in the parameter space. This initial direction determines the form of V. If the trajectory corresponds to asymptotic freedom, i.e., if the Gaussian fixed point appears as an ultraviolet fixed point on the trajectory, the potential will grow to engender a universe. A trajectory that is non-free asymptotically is a critical line on which all points are equivalent to the fixed point, and the system behaves as if it had never left the fixed point, with the time development as described in Appendix A.

All quantities with dimension scale with appropriate powers of Λ. The potential V is of dimensionality (length)$^{-4}$, and we introduce a dimensionless potential U by writing

$$V = \Lambda^4 U \tag{16}$$

Under a scale transformation, U changes under renormalization according to

$$\Lambda \frac{\partial U}{\partial \Lambda} = \beta\left[U\right] \tag{17}$$

where the "beta-function" $\beta\left[U\right]$ is a functional of U. Near the Gaussian fixed point, where $U \equiv 0$, we can make a linear approximation

$$\beta\left[U\right] \approx -bU \tag{18}$$

leading to an eigenvalue equation

$$\Lambda \frac{dU_b}{d\Lambda} = -bU_b \tag{19}$$

which defines the eigenpotential U_b. In the linear approximation, the most general U is a linear superposition of these eigenpotentials.

From the renormalization-group analysis briefly summarized in Appendix B, one obtains the solution

$$\begin{aligned} U_b(z) &= c\Lambda^{-b}\left[M\left(-2+b/2, N/2, z\right) - 1\right] \\ z &= 8\pi^2 \sum_n \varphi_n^2 \end{aligned} \tag{20}$$

where M is a Kummer function, c is an arbitrary constant, and $\varphi_n\left(x\right)$ is a dimensionless field:

$$\varphi_n\left(x\right) = \frac{\hbar}{\Lambda}\phi_n\left(x\right) \tag{21}$$

Again, we restore units to remind us that the potential depends on $\hbar$.

The power series and asymptotic behavior of the Kummer function are given by

$$
\begin{aligned}
M(p,q,z) &= 1+\frac{p}{q}z+\frac{p\,(p+1)}{q\,(q+1)}\frac{z^2}{2!}+\frac{p\,(p+1)\,(p+2)}{q\,(q+1)\,(q+2)}\frac{z^3}{3!}+\cdots \\
M(p,q,z) &\approx \Gamma\,(q)\,\Gamma^{-1}\,(p)\,z^{p-q}\exp z
\end{aligned}
\tag{22}
$$

Using the derivative formula [7]

$$
M'\,(p,q,z)=pq^{-1}M\,(p+1,q+1,z) \tag{23}
$$

we obtain

$$
U_b'(z)=-c\Lambda^{-b}N^{-1}\,(4-b)\,M\,(-1+b/2,1+N/2,z) \tag{24}
$$

Asymptotic freedom corresponds to $b>0$, and spontaneous symmetry breaking occurs when $b<2$. Thus we limit ourselves to the range $0<b<2$.

The limiting case $b=2$ corresponds to the massive free field, which is asymptotically free but does not exhibit spontaneous symmetry breaking, i.e., it does not maintain a vacuum field. The limiting case $b=0$ corresponds to the ϕ^4 theory, which exhibits spontaneous symmetry breaking, but is not asymptotically free. In our linear approximation (19), $b=0$ corresponds to $\Lambda\partial U/\partial\Lambda=0$, which indicates neutrality. However, the beta-function to second order gives [8]

$$
\Lambda\frac{\partial U}{\partial\Lambda}=\frac{3}{16\pi^2}U^2 \qquad \left(\text{for } b=0,\text{ or } \phi^4 \text{ theory}\right) \tag{25}
$$

which shows it increases as Λ increases, and is thus asymptotically non-free.

We emphasize that the HH eigenpotential is derived (a) in flat space-time, (b) in the neighborhood of the Gaussian fixed point, where $U\equiv 0$. Corrections due to space-time curvature and nonlinearity in U have not been calculated; but the present approximation should be good in a neighborhood of the big bang.

IV. COSMOLOGICAL EQUATIONS

The canonical Lagrangian (14) of the scalar field gives the following equation of motion and components of the energy-momentum tensor :

$$\begin{aligned}\ddot{\phi}_n &= -3H\dot{\phi}_n - \frac{\partial V}{\partial \phi_n} \\ \rho_{\text{canon}} &= \frac{1}{2}\sum_{n=1}^{N}\dot{\phi}_n^2 + V \\ p_{\text{canon}} &= \frac{1}{2}\sum_{n=1}^{N}\dot{\phi}_n^2 - V\end{aligned} \tag{26}$$

The constraint equation (9) now reads

$$X \equiv H^2 + \frac{k}{a^2} - \frac{1}{3}\sum_n \dot{\phi}_n^2 - \frac{2}{3}V = 0 \tag{27}$$

On general principle, the equations of motion must guarantee $\dot{X} = 0$, since it is known that the Cauchy problem in general relativity exists [9]. However, direct computation using X as given in (27) yields $\dot{X} = -(2/3)\,\dot{a}\,(\partial V/\partial a)$, which is nonzero if the cutoff depends on the time. This defect can be attributed to the fact that the gravitational cutoff has not been built into the Lagrangian (14). As remedy, we modify $T^{\mu\nu}$ of the scalar field by adding a term to the pressure, and take

$$\begin{aligned}\rho &= \rho_{\text{canon}} \\ p &= p_{\text{canon}} - \frac{a}{3}\frac{\partial V}{\partial a}\end{aligned} \tag{28}$$

For an eigenpotential $V = a^{-4}U_b$ it can shown that

$$a\frac{\partial V}{\partial a} = (b-4)V + \sum_n \phi_n \frac{\partial V}{\partial \phi_n} \tag{29}$$

The cosmological equations now become

$$\begin{aligned}\dot{H} &= \frac{k}{a^2} - \sum_n \dot{\phi}_n^2 + \frac{1}{3}a\frac{\partial V}{\partial a} \\ \ddot{\phi}_n &= -3H\dot{\phi}_n - \frac{\partial V}{\partial \phi_n} \\ X &\equiv H^2 + \frac{k}{a^2} - \frac{1}{3}\sum_n \dot{\phi}_n^2 - \frac{2}{3}V = 0\end{aligned} \tag{30}$$

The first two equations now imply $\dot{X} = 0$, and we have a closed set of self-consistent equations.

We are able to work with a set of classical equations, because we have neglected quantum fluctuations about the vacuum scalar field. However, important quantum effects are incorporated through the scale dependence of the potential V arising from renormalization.

V. CONSTRAINT EQUATION AND POWER LAW

The constraint equation in (30) requires

$$H = \left(\frac{2}{3} V + \frac{1}{3} \sum_n \dot{\phi}_n^2 - \frac{k}{a^2} \right)^{1/2} \tag{31}$$

That H be real and finite imposes severe restrictions on initial values. In particular, $a = 0$ is ruled out; the initial state cannot be exactly at the big bang. This poses no problem from a practical point of view, for an initial universe with radius $a \sim 1$ (Planck units) is practically a point.

From a physical point of view, we do not expect the model to be valid in the immediate neighborhood of the big bang, which would be dominated by quantum fluctuations. The universe could have been created at very high temperatures, and rapidly cooled through a phase transition to reach a vacuum with spontaneous broken symmetry. Or it could have been be created in the broken state. There is no way to know what actually happened; all we know is that we start our model at some time after the big bang, but still in the Planck era, with a vacuum field already present.

Now we turn to the consequence of the constraint. Since $V = a^{-4} U$, it would vanish rather rapidly in an expanding universe. The same is true of ϕ_n, which is proportional to a^{-1} by dimension analysis. Thus, the constraint (31) would make $H \to 0$. Given the absence of relevant scale, we expect H to obey a power law:

$$\begin{aligned} H &\sim t^{-p} \\ a &\sim \exp t^{1-p} \end{aligned} \tag{32}$$

The argument is far from rigorous, of course, but the result is support by the exactly solution for the massless free field (Appendix A), and is verified in numerical solutions to

be discussed. The latter show that the power law emerges after averaging over small high-frequency oscillations.

VI. NUMERICAL SOLUTIONS

For numerical solutions, we limit ourselves to the simplest case, a real scalar field ($N = 1$). A multi-component field would yield qualitatively the same results for a completely uniform universe. It is convenient to rewrite the cosmological equations as a set of first-order autonomous equations:

$$\begin{aligned}
\dot{a} &= Ha \\
\dot{H} &= \frac{k}{a^2} - v^2 + \frac{1}{3}a\frac{\partial V}{\partial a} \\
\dot{\phi} &= v \\
\dot{v} &= -3Hv - \frac{\partial V}{\partial \phi}
\end{aligned} \tag{33}$$

There are 4 unknown functions of time: a, H, ϕ, v. The initial values must be real, and satisfy the constraint

$$H = \left(\frac{2}{3}V + \frac{1}{3}\dot{\phi}^2 - \frac{k}{a^2}\right)^{1/2} \tag{34}$$

Although this relation is preserve by the equations, numerical procedures tend to violate it, and it is difficult to extend time iterations indefinitely. As an exploratory investigation, we have not looked into algorithm improvement.

For completeness, we restate the HH potential V, which is generally a linear superposition of eigenpotentials V_b:

$$\begin{aligned}
V_b(\phi) &= a^{-4}U_b(z) \\
U_b(z) &= ca^b\left[M\left(-2+b/2, 1/2, z\right) - 1\right] \\
z &= 8\pi^2 a^2 \phi^2
\end{aligned} \tag{35}$$

where M is the Kummer function. Some useful formulas are

k	b	c	a_0	ϕ_0	$\dot{\phi}_0$	H_0	p
-1	1	0.1	1.00	0.01	0.1	1.00	0.81
0	1	0.1	1.85	0.17	0.2	0.91	0.65
1	1	0.1	1.85	0.19	0.2	1.70	0.15

TABLE I. Computation data: k = curvature; b,c = potential parameters; others = initial data; p = output exponent.

$$\begin{aligned} a\frac{\partial V_b}{\partial a} &= (b-4)V_b + \phi\frac{\partial V_b}{\partial \phi} \\ \frac{\partial V_b}{\partial \phi} &= 16\pi^2 a^{-2}\phi U_b' \\ U_b'(z) &= -c\,(4-b)\,a^b M\,(-1+b/2, 3/2, z) \end{aligned} \tag{36}$$

The model parameters are

$$\begin{aligned} &\text{Curvature:} && k = 1, 0, -1 \\ &\text{Eigenvalue:} && 0 < b < 2 \\ &\text{Potential strength:} && c \end{aligned} \tag{37}$$

A pair of values $\{b, c\}$ should be specified for each eigenpotential in V. The c's should be real numbers of either sign, such that V be positive for large ϕ, and have a lowest minimum at $\phi \neq 0$.

First we use an eigenpotential with $b = 1$, which is shown in Fig.1 at $a = 1$. As the universe expands, it will increase uniformly by a factor $a(t)$. This property is a linear approximation that holds for sufficiently small $a(t)$. Fig.2 shows numerical results for this potential, for curvature parameter $k = 0$. We see that $H(t)$ oscillates about an average behavior consilient with a power law $H \sim t^{-p}$, with $p = 0.65$. The main source of uncertainty in p arises from the limitation on time iterations, due to numerical violation of the constraint. Numerical results for p from a number of runs are tabulated in Table I.

Next we consider a superposition of two eigenpotentials:

$$\begin{aligned} U(z) &= c_1 U_{b_1}(z) - c_2 U_{b_2}(z) \\ b_1 = 1.6,&\qquad c_1 = 0.1 \\ b_2 = 0.4,&\qquad c_2 = 5.0 \end{aligned} \tag{38}$$

The locations $\pm z_{\min}$ and the depth $U_{\min}$ of the minima are functions of a, and are plotted in Fig.3. Because of the large ratio $c_2/c_1 = 50$, $U_{\min}$ suddenly jumps at a near-critical value $a_c \approx 5$. For $a < a_c$, the minima of the can be approximated by two symmetrically placed δ-functions; the scalar field becomes trapped at values $\pm\phi_1$ corresponding to the minima, and the model approaches the Ising spin model. Results of numerical solutions are shown in Fig. 4, with curvature parameter is $k = 0$, and the initials conditions are $a_0 = 1, \phi_0 = 0, \dot{\phi}_0 = 0.1$.

Figs. 2 and 4 show that the scalar field oscillates during cosmic expansion, contrary to the "slow-roll" picture of inflation. Closer examination show that the oscillation amplitudes are so large as to sample the exponential region of the potential wall. That is, the distinctive part of the HH potential, which makes it asymptotically free, plays an important role in the expansion of the universe.

VII. COMPARISON WITH OBSERVATIONS

Our model is valid only in the Planck era, and does not contain matter apart from the vacuum scalar field. We shall nevertheless compare the model with present observations, assuming that the power law $H(t) \sim h_0 t^{-p}$ will persist in the real universe. The index p depends on model parameters, which might change with conditions in the universe such as the temperature. For our analysis, however, we take p to be a constant. All quantities are measured in Planck units, unless otherwise specified.

The age of the universe t_0 and the present value $H_{\text{now}} = H(t_0)$ are taken to be

$$\begin{aligned} t_0 &= 1.5 \times 10^{10} \text{ yrs} \approx 10^{60} \\ H_{\text{now}} &= t_0^{-1} \end{aligned} \tag{39}$$

The initial value, defined at $t = 1$, is given by

$$H_{\text{initial}} = h_0 \left(1.65 \times 10^{50}\right)^{-(1-p)} \tag{40}$$

If we put $H_{\text{initial}} = 1$ as a natural value, then h_0 gives the fine-tune factor, which are tabulated Table I.

The radius of the universe expands according to

$$a(t) = a_0 \exp \frac{h_0 t^{1-p}}{1-p}$$

p	h_0
0.5	1.25×10^{25}
0.85	3×10^7
0.95	300
0.99	3

TABLE II. Fine-tune factor for Hubble's parameter

p	a_{now}/a_0
0.5	7.4
0.85	786
0.95	5×10^8
0.99	3×10^{43}

TABLE III. Present radius of universe

The present radius is $a(1)$:

$$a_{\text{now}} = a_0 \exp \frac{1}{1-p} \tag{41}$$

Some values are tabulated in Table II.

Under the assumption that p is constant, its most reasonable value would lie in the range $0.99 < p < 1$.

We now turn to the galactic redshift. The relation between the luminosity distance d_L of the source and the redshift parameter z is implicitly given by the following relations [10]:

$$\begin{aligned} z &= \frac{a(t_0)}{a(t_1)} - 1 \\ f(r_1) &= \int_{t_1}^{t_0} \frac{dt}{a(t)} \\ d_L &= \frac{r_1 a^2(t_0)}{a(t_1)} = r_1 a(t_0)(1+z) \end{aligned} \tag{42}$$

where t_0 the the time of detection, at the origin of the coordinate system, of light emitted at time $t_1 < t_0$, by a source located at co-moving coordinate r_1. The function f is defined

by

$$f\left(r_1\right) \equiv \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} \sin^{-1} r_1 & (k=1) \\ r_1 & (k=0) \\ \sinh^{-1} r_1 & (k=-1) \end{cases} \tag{43}$$

Using the first two equations, we can expressed r_1 and t_1 in terms of t_0 and z, and then obtain $d_L\left(z\right)$ from the third equation.

In our model, $a\left(t\right) = a_0 \exp\left(\xi t^{1-p}\right)$,where $\xi = h_0 \left(1-p\right)^{-1}$. Define an effective time $\tau = \xi t^{1-p}$. For $0 < p < 1$, the second equation in (42) can be rewritten as

$$f\left(r_1\right) = K_0 \int_{\tau_1}^{\tau_0} d\tau \tau^{p/(1-p)} \exp\left(-\tau\right) \tag{44}$$

where $K_0 = \left[\left(1-p\right) a_0\right]^{-1} \xi^{-1/(1-p)}$, and

$$\begin{aligned} \tau_0 &= \xi t_0^{1-p} \\ \tau_1 &= \tau_0 - \ln\left(z+1\right) \end{aligned} \tag{45}$$

Since $t_0 \approx 10^{60}$, we can assume $\tau_0 >> 1$, and obtain to a good approximation $f\left(r_1\right) \approx K_1 z$,where $K_1 = K_0 \tau_0^{p/(1-p)} \exp\left(-\tau_0\right)$. Since K_0 is extremely small, this gives $r_1 = z$ to a very good approximation, and thus

$$d_L = K_1 a_0 z \left(1+z\right) \tag{46}$$

We rewrite this as

$$\frac{d_L}{z} = d_0 \eta \left(1+z\right) \tag{47}$$

where $d_0 = c/H_{\text{now}} = 4283$ Mpc, corresponding to the choice $H_{\text{now}} = 70$ km s^{-1}Mpc^{-1}.

Fig.5 shows comparison with data from observations and supernovas [11] and gamma-ray bursts [12]. The upper panel shows the parameter μ used in conventional data analysis:

$$\mu = 5 \log\left(\frac{d_L}{\text{Mpc}}\right) + 25 \tag{48}$$

plotted as a function of z. The lower panel shows a semilog plot of d_L/z vs. z. Lines corresponding to Hubble's law (no dark energy) are shown. The p-dependence affects only the vertical displacement but not the shape of the model curves. Curve A corresponds to (47) with $\eta = 1$, and curve B with $\eta = 1/4$. Curve A fits the data for $z < 1$, while curve B could represent the situation in a large-z regime beyond present measurements.

The power-law model allows only for variations in d_0, which may come from variations in the exponent p, caused by conditions such as the temperature. This leads us to speculate that the universe may have had gone through a broad phase transition, or crossover, connecting two situations corresponding respectively to the curves A and B. The transition was completed around $z = 1$.

The relation between the emission time and the red shift can be obtained from (45):

$$\frac{t_1}{t_0} = [1 - (1-p)\ln(z+1)]^{1/(1-p)} \tag{49}$$

For $p \approx 1$, we put $p = 1 - \epsilon$ and obtain

$$\frac{t_1}{t_0} \approx [1 - \epsilon \ln(z+1)]^{1/\epsilon} \underset{\epsilon \to 0}{\longrightarrow} (z+1)^{-1} \tag{50}$$

Assuming this relation, we judge that the transition was completed at $t_1/t_0 \approx 0.5$, or about than 7 billion years ago.

VIII. COSMIC INFLATION AND DECOUPLING

The problem of cosmic inflation is inseparable from that of matter creation, which has not been taken into account in our model so far. Most of the matter in the universe should have been created by the end of the inflation era, in order that the memory of the original density be imprinted.

An equally important problem relates to energy scales. Our equations so far contains only one scale, the Planck scale. With matter creation, there emerges the scale of nuclear interactions, which is smaller than the Planck scale by some 18 orders of magnitude. Physically speaking, the matter scale emerges in QCD spontaneously through "dimensional transmutation" [13]. In our model, we would introduce it through the coupling parameter between the scalar field and matter. These two scales must eventually decouple from each other. That is, the cosmological equations should break up into two sets, one governing the expansion, the other galactic evolution, and in each set the information about the other set occurs only through lumped constants. What is the mechanism for this decoupling?

To address these questions, we model matter as a perfect fluid coupled to the scalar field, and obtain a set of cosmological equations that, again, represent an initial-value problem. These are derived in Appendix C. Numerical studies of these equations, both for a real scalar

field and a complex scalar field, lead us to the conclusion that a completely uniform scalar field, even with more than one components, cannot create sufficient matter to satisfy the inflation scenario. More important, it cannot exhibit the decoupling desired.

We are led to an attempt to relax complete uniformity, within the dictate of the RW metric. It is natural to consider a complex scalar field with uniform modulus, but spatially varying phase. The phase variation gives rise to superfluid velocity, with the attendant vortex dynamics. The universe then becomes a superfluid with vortex dynamics. New physics emerges, namely the growth and decay of a vortex tangle that fills the universe, signifying quantum turbulence. We find that this provides a framework for matter creation, and the decoupling of scales.

In the extension of our model, the demise of quantum turbulence will signifies the end of the inflation era, as well as the validity of our model, for density fluctuations would become important. The standard hot big bang theory will then take over, with one addition: the universe remains a superfluid with vorticity. The latter will offer explanations to post-inflation phenomena such as galactic voids, galactic jets, and the dark mass. We will present this development in detail in paper II of this series [5].

Appendix A: The massless free field

The cosmological equations with a real massless scalar field, corresponding to $V \equiv 0$, are

$$\begin{aligned}
\dot{a} &= Ha \\
\dot{H} &= \frac{k}{a^2} - \dot{\phi}^2 \\
\ddot{\phi} &= -3H\dot{\phi} \\
X \equiv H^2 - \frac{1}{3}\dot{\phi}^2 + \frac{k}{a^2} &= 0
\end{aligned} \tag{A1}$$

They describe what happens if the scalar field remains at the Gaussian fixed point. The last equation $X = 0$ is the constraint equation, and X is a constant of the motion.

The third equation can be rewritten in the form $d\ln\left(\dot{\phi}a^3\right)/dt = 0$, which gives

$$\dot{\phi} = c_0 a^{-3} \tag{A2}$$

where c_0 is an arbitrary constant. The equations then reduce to

$$\begin{aligned}\dot{a} &= Ha \\ \dot{H} &= \frac{k}{a^2} - \frac{c_0^2}{a^6} \\ H^2 &= \frac{c_1}{a^6} - \frac{k}{a^2}\end{aligned} \tag{A3}$$

where $c_1 = c_0^2/3$. Dividing the second equation by the first, and equating $\dot{H}/\dot{a} = dH/da$,we obtain

$$HdH = \left(\frac{k}{a^3} - \frac{c_0^2}{a^7}\right) da \tag{A4}$$

Integrating both sides gives

$$H = \pm\sqrt{\frac{c_1}{a^6} + c_2 - \frac{k}{a^2}} \tag{A5}$$

Since $H = \dot{a}/a$, this can be further integrated to yield

$$t = \pm \int \frac{da}{\sqrt{c_1 a^{-4} + c_2 a^2 - k}} \tag{A6}$$

where c_2 is an arbitrary constant. The $\pm$ signs reflect the time-reversal invariance of the equations. We choose the positive sign to obtain

$$a\left(t\right) \underset{t\to\infty}{\longrightarrow} a_0 \exp\left(\sqrt{c_2} t\right) \tag{A7}$$

This is the general solution without constraint, and c_2 is the equivalent cosmological constant.

The constraint equation can be put in the form

$$\frac{\dot{a}}{a} = \pm\sqrt{c_1 a^{-6} - k a^{-2}} \tag{A8}$$

which gives

$$t = \pm \int \frac{da}{\sqrt{c_1 a^{-4} - k}} \tag{A9}$$

Comparison with (**??**) shows

$$c_2 = 0 \tag{A10}$$

Thus, (A7) is incorrect; the constraint "fine-tunes" the cosmological constant to zero. The correct solution gives

$$a(t) \begin{cases} = c_1^{-1/6} t^{1/3} & (k = 0) \\ \underset{t\to\infty}{\longrightarrow} c_1^{-1/4} & (k = 1) \\ \underset{t\to\infty}{\longrightarrow} t & (k = -1) \end{cases} \tag{A11}$$

which corresponds to a power-law

$$H \underset{t\to\infty}{\longrightarrow} h_0 t^{-1} \tag{A12}$$

Appendix B: Renormalization and the Halpern-Huang potential

A distinctive feature of quantum field theory is that the field can propagate virtually. This is described by the propagator function, which for a free field has Fourier transform $\Delta\left(k^2\right) = k^{-2}$. The high-$k$, or high-energy modes must be cut off, for otherwise the virtual processes lead to divergences, rendering the quantum theory meaningless. The cut off energy Λ is introduced by "regulating" the propagator:

$$\begin{aligned} \Delta\left(k^2\right) &= \frac{f(k^2/\Lambda^2)}{k^2} \\ f\left(z\right) &\underset{z\to\infty}{\rightarrow} 0 \end{aligned} \tag{B1}$$

The detailed form of $f(k^2/\Lambda^2)$ is not important. What is important is that Λ is the only scale in the theory. The regulated propagator in configurational space will be denoted by $K\left(x,\Lambda\right)$.

In the formulation of renormalization according to Wilson [14,15], interaction coupling parameters must change with Λ, in such a fashion as to preserve the theory. This is called "renormalization". For a given value of Λ, the parameters define an effective theory appropriate to that energy scale. A reformulation of the Wilson scheme using functional methods has been given by Polchinski [16].

Interactions that go to zero in the short-distance limit (or infinite-energy limit) are said to be asymptotically free, an example of which is the gauge interaction in QCD. In the opposite non-free behavior, the interactions grow indefinitely with decreasing length scale, and would diverge in the limit. This is the behavior found in QED and the ϕ^4 scalar field, for which the short-distance limit can exist only if there is no interaction at all. For applications in cosmology, we want interactions that vanish at the big bang, the small-distance limit, which means asymptotically free interactions.

The Halpern-Huang (HH) potential was originally derived [4] by summing one-loop Feynman graphs. Here we outline an improved derivation due to Periwal [17], which is based on Polchinski's functional method of renormalization. For simplicity consider a real scalar field

($N=1$). The action in d-dimensional Euclidean space-time can be written as

$$S[\phi,\Lambda]=S_0[\phi,\Lambda]+S'[\phi,\Lambda] \tag{B2}$$

where the first term corresponds to the free field, and the second term represents the interaction. We have

$$S_0\left[\phi,\Lambda\right]=\frac{1}{2}\int d^dxd^dy\,\phi\left(x\right)K^{-1}\left(x-y,\Lambda\right)\phi\left(y\right) \tag{B3}$$

where $K^{-1}\left(x-y,\Lambda\right)$ is the inverse of the propagator $K(x-y,\Lambda)$, in an operator sense. It differs from the Laplacian operator significantly only in a neighborhood of $|x-y|=0$, of radius Λ^{-1}. The partition function with external source J, which generates all correlation functions of the theory, is given by

$$Z[J,\Lambda]=\mathcal{N}\int D\phi e^{-S[\phi,\Lambda]-(J,\phi)} \tag{B4}$$

where $\mathcal{N}$ is a normalization constant, which may depend on Λ, and $(J,\phi)=\int d^dxJ\left(x\right)\phi\left(x\right)$.

In Wilson's renormalization scheme, modes contributing to the integral in (B4) with momentum higher than Λ are "integrated out", but not discarded, in order to lower the effective cutoff. This leads to a change the form of S', but the system itself is unaltered. The interactions are then said to be "renormalized". In a general sense, renormalization means changing the cutoff Λ with simultaneous change in the form of S', so as to leave Z invariant, i.e.,

$$\frac{dZ[J,\Lambda]}{d\Lambda}=0 \tag{B5}$$

This constraint is solved by Polchinski's renormalization equation,which is a functional integro-differential equation for $S'\left[\phi,\Lambda\right]$. For $J\equiv 0$, it reads

$$\frac{dS'}{d\Lambda}=-\frac{1}{2}\int dxdy\frac{\partial K\left(x-y,\Lambda\right)}{\partial\Lambda}\left[\frac{\delta^2S'}{\delta\phi\left(x\right)\delta\phi\left(y\right)}-\frac{\delta S'}{\delta\phi\left(x\right)}\frac{\delta S'}{\delta\phi\left(y\right)}\right] \tag{B6}$$

Assuming that there are no derivative couplings, we can write S' as the integral of a local potential:

$$\begin{aligned}S'\left[\phi,\Lambda\right]&=\Lambda^d\int d^dxU\left(\varphi\left(x\right),\Lambda\right)\\ \varphi\left(x\right)&=\Lambda^{1-d/2}\phi\left(x\right)\end{aligned} \tag{B7}$$

where U is a dimensionless function, and φ is a dimensionless field. In the neighborhood of the Gaussian fixed point, where $S' = 0$, we can linearize (B6) by neglecting the last term, and obtain a linear differential equation for $U(\varphi, \Lambda)$:

$$\Lambda \frac{\partial U}{\partial \Lambda} + \frac{\kappa}{2} U'' + \left(1 - \frac{d}{2}\right) \varphi U' + Ud = 0 \tag{B8}$$

where a prime denote partial derivative with respect to φ, and $\kappa = \Lambda^{3-d} \partial K(0, \Lambda) / \partial \Lambda$. Now we seek eigenpotentials $U_b(\varphi, \Lambda)$ with the property

$$\Lambda \frac{\partial U_b}{\partial \Lambda} = -b U_b \tag{B9}$$

In the language of perturbative renormalization theory, the right side is the linear approximation to the β-function. Substituting this into the previous equation, we obtain the differential equation

$$\left[\frac{\kappa}{2} \frac{d^2}{d\varphi^2} - \frac{1}{2}(d-2) \varphi \frac{d}{d\varphi} + (d-b)\right] U_b = 0 \tag{B10}$$

Since this equation does not depend on Λ, the Λ-dependence of the potential is contained in a multiplicative factor. In view of (B9), the factor is Λ^{-b}.

For $d \neq 2$, (B10) can be transformed into Kummer's equation:

$$\left[z \frac{d^2}{dz^2} + (q - z) \frac{d}{dz} - p\right] U_b = 0 \tag{B11}$$

where

$$\begin{aligned} q &= 1/2 \\ p &= \frac{b-d}{d-2} \\ z &= (2\kappa)^{-1} (d-2) \varphi^2 \end{aligned} \tag{B12}$$

The solution is

$$U_b(z) = c \Lambda^{-b} \left[M(p, q, z) - 1\right] \tag{B13}$$

where c is an arbitrary constant, and M is the Kummer function. We have subtracted 1 to make $U_b(0) = 0$. This is permissible, since it merely changes the normalization of the partition function. In (20), the value of κ corresponds to a sharp cutoff.

For $d = 2$, the solution to (B10) is sinusoidal, and the theory reduces to the XY model, or equivalently the so-called sine-Gordon theory [18].

Appendix C: Coupling to perfect fluid

We discuss how the cosmological equations (30) may be generalized to include coupling to galactic matter modeled as a perfect fluid, whose energy-momentum tensor is given by [19]

$$T_{\mathrm{m}}^{\mu\nu} = -g^{\mu\nu}\rho_{\mathrm{m}} + (p_{\mathrm{m}} + \rho_{\mathrm{m}})\, U^{\mu}U^{\nu} \tag{C1}$$

where ρ_{m} is the energy density, and U^{μ} is a velocity field, with $g_{\mu\nu}U^{\mu}U^{\nu} = 1$. For a spatially uniform fluid, $U^{0} = 1$, $U^{j} = 0$. We assume the equation of state

$$p_{\mathrm{m}} = \epsilon_0 \rho_{\mathrm{m}} \tag{C2}$$

where $\epsilon_0 = 1/3$ for radiation, and $\epsilon_0 = 0$ for classical matter. The coupling to the scalar field is specified via an interaction Lagrangian density $\mathcal{L}_{\mathrm{int}}$. We give some examples of possible interactions.

The simplest interaction is a direct interaction with a real scalar field: $\mathcal{L}_{\mathrm{int}} = -\lambda\rho_{\mathrm{m}}\phi$. Current-current interaction with a complex scalar field ($N = 2$) can be constructed as follows. Represent the scalar field in terms of $\phi = 2^{-1/2}(\phi_1 + i\phi_2)$ and its complex conjugate ϕ^*, or in terms of the phase representation $\phi = F\exp(i\sigma)$. The conserved scalar current density in the absence of interaction is $J_{\mu}^{\mathrm{sc}} = (2i)^{-1}(\phi^*\partial_{\mu}\phi - \phi\partial_{\mu}\phi^*) = F^2\partial_{\mu}\sigma$. The current density of a perfect fluid is $J_{\nu}^{\mathrm{m}} = \rho_{\mathrm{m}}U_{\nu}$. The current-current interaction corresponds to

$$\begin{aligned}\mathcal{L}_{\mathrm{int}} &= -\lambda g^{\mu\nu}J_{\mu}^{\mathrm{sc}}J_{\nu}^{\mathrm{m}} = \lambda\rho_{\mathrm{m}}g^{\mu\nu}F^2(\partial_{\mu}\sigma)\,U_{\nu} \\ &= -\lambda\rho_{\mathrm{m}}F^2\dot{\sigma} \qquad \text{(spatially uniform system)}\end{aligned} \tag{C3}$$

Returning to the general case, we can decompose the total energy-momentum tensor of scalar field and perfect fluid as follows:

$$T^{\mu\nu} = T_{\mathrm{sc}}^{\mu\nu} + T_{\mathrm{m}}^{\mu\nu} + T_{\mathrm{int}}^{\mu\nu} \tag{C4}$$

We assume

$$T_{\mathrm{int}}^{\mu\nu} = -g^{\mu\nu}\mathcal{L}_{\mathrm{int}} \tag{C5}$$

which leads to an interaction energy density ρ_{int} and pressure p_{int}:

$$\begin{aligned}\rho_{\mathrm{int}} &= -\mathcal{L}_{\mathrm{int}} \\ p_{\mathrm{int}} &= \mathcal{L}_{\mathrm{int}}\end{aligned} \tag{C6}$$

The equation of motion for the perfect fluid comes from the conservation law $T^{\mu\nu}_{;\mu} = 0$, which for a spatially uniform system reduces to

$$\dot{\rho} + 3H\left(\rho + p\right) = 0 \tag{C7}$$

where

$$\begin{aligned} \rho &= \rho_{\rm sc} + \rho_{\rm m} + \rho_{\rm int} = \frac{1}{2}\sum_n \dot{\phi}_n^2 + V \;\; + \rho_{\rm m} + \mathcal{L}_{\rm int} \\ p &= p_{\rm sc} + p_{\rm m} + p_{\rm int} = \frac{1}{2}\sum_n \dot{\phi}_n^2 - V \;\; + \epsilon_0\rho_{\rm m} - \mathcal{L}_{\rm int} \end{aligned} \tag{C8}$$

We can rewrite (C7) in a more useful form. First, multiply both sides of the ϕ_n equation in (30) by $\dot{\phi}_n$:

$$\begin{aligned} \dot{\phi}_n\ddot{\phi}_n &= -3H\dot{\phi}_n^2 - \frac{\partial V}{\partial \phi_n}\dot{\phi}_n + \frac{\partial \mathcal{L}_{\rm int}}{\partial \phi_n}\dot{\phi}_n \\ \frac{1}{2}\frac{d}{dt}\sum_n \dot{\phi}_n^2 &= -3H\sum_n \dot{\phi}_n^2 - \sum_n \frac{\partial V}{\partial \phi_n}\dot{\phi}_n + \sum_n \frac{\partial \mathcal{L}_{\rm int}}{\partial \phi_n}\dot{\phi}_n \end{aligned} \tag{C9}$$

We write

$$\frac{dV}{dt} = \sum_n \frac{\partial V}{\partial \phi_n}\dot{\phi}_n + \frac{\partial V}{\partial \Lambda}\dot{\Lambda} \tag{C10}$$

Thus

$$\sum_n \frac{\partial V}{\partial \phi_n}\dot{\phi}_n = \frac{dV}{dt} - \frac{\partial V}{\partial \Lambda}\dot{\Lambda} \tag{C11}$$

Using this we get

$$\frac{d}{dt}\left(\frac{1}{2}\sum_n \dot{\phi}_n^2 + V\right) = -3H\sum_n \dot{\phi}_n^2 + \sum_n \frac{\partial \mathcal{L}_{\rm int}}{\partial \phi_n}\dot{\phi}_n + a\frac{\partial V}{\partial a}H \tag{C12}$$

Now, using (C8), we can rewrite (C7) as

$$\frac{d}{dt}\left[\frac{1}{2}\sum_n \dot{\phi}_n^2 + V \;\; + \rho_{\rm m} + \mathcal{L}_{\rm int}\right] = -3H\left[\frac{1}{2}\sum_n \dot{\phi}_n^2 + \left(1 + \epsilon_0\right)\rho_{\rm m}\right] \tag{C13}$$

Using the equation before this, we finally obtain

$$\frac{d\rho_{\rm m}}{dt} = -3H\left(1 + \epsilon_0\right)\rho_{\rm m} - \sum_n \frac{\partial \mathcal{L}_{\rm int}}{\partial \phi_n}\dot{\phi}_n - \frac{d\mathcal{L}_{\rm int}}{dt} - a\frac{\partial V}{\partial a}H \tag{C14}$$

In summary, the cosmological equations are, with $H = \dot{a}/a$,

$$
\begin{aligned}
\dot{H} &= \frac{k}{a^2} - 4\pi G\left[\sum_n \dot{\phi}_n^2 + (1+\epsilon_0)\,\rho_{\mathrm{m}}\right] + \frac{1}{3}a\frac{\partial V}{\partial a} \\
\ddot{\phi}_n &= -3H\dot{\phi}_n - \frac{\partial V}{\partial \phi_n} + \frac{\partial \mathcal{L}_{\mathrm{int}}}{\partial \phi_n} \\
\dot{\rho}_{\mathrm{m}} &= -3H\,(1+\epsilon_0)\,\rho_{\mathrm{m}} - \sum_n \frac{\partial \mathcal{L}_{\mathrm{int}}}{\partial \phi_n}\dot{\phi}_n - \frac{d\mathcal{L}_{\mathrm{int}}}{dt} - Ha\frac{\partial V}{\partial a} \\
H^2 &= \frac{2}{3}\left(\frac{1}{2}\sum_{n=1}^{N}\dot{\phi}_n^2 + V + \rho_{\mathrm{m}}\right) - \frac{k}{a^2}
\end{aligned}
\tag{C15}
$$

The last equation is a constraint on initial conditions, and is preserved by the equations of motion. This defines a self-consistent initial-value problem.

Analytical and numerical studies show that matter creation is inefficient, and that no decoupling occurs between expansion and matter dynamics.

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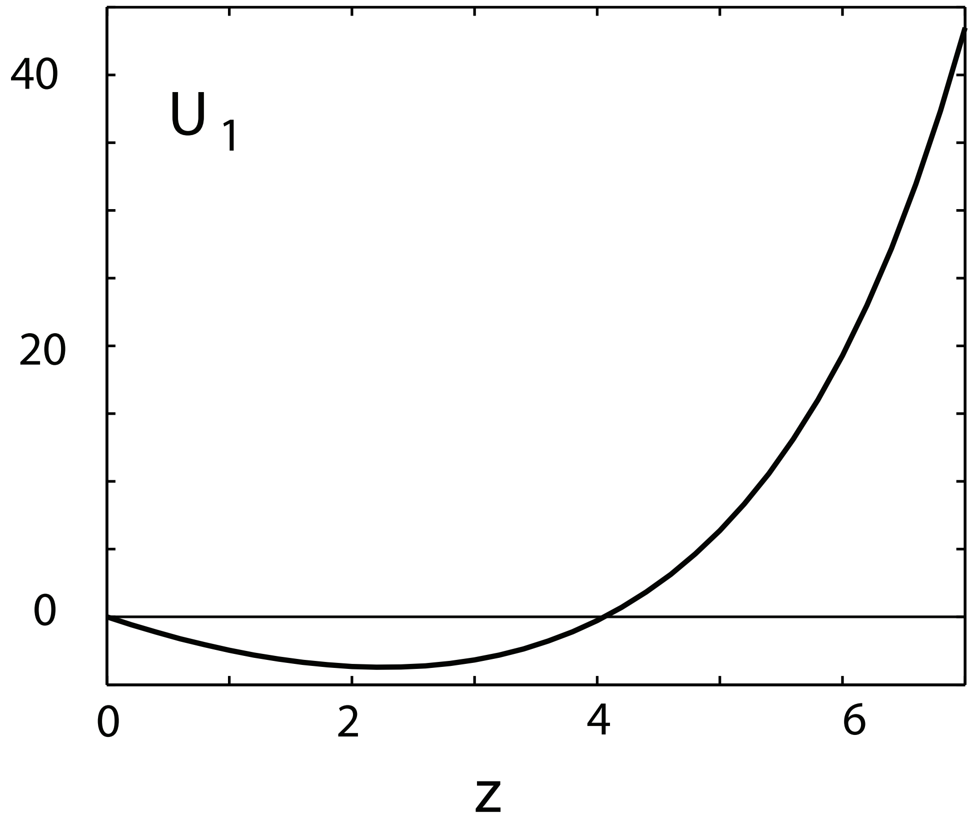

FIG. 1. The Halpern-Huang eigenpotential $U_1(z)$, with $z = 8\pi^2\,(a\phi)^2$, where ϕ is a real scalar field, and a is the Robertson-Walker length scale. The potential increases exponentially for large z.

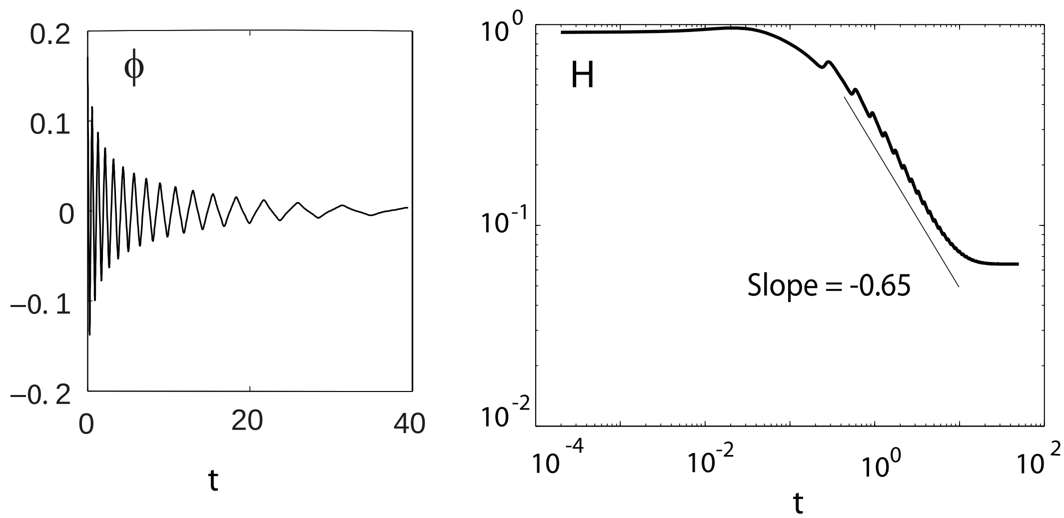

FIG. 2. Results from solving the initial-value problem with the potential U_1 of Fig.1. The Hubble parameter H follows a power law t^{-p} after averaging over small oscillations. The flat tail is spurious, arising from numerical instability. The scalar field ϕ oscillates with large amplitudes, sampling the exponential region of the potential. The behavior is quite different from the "slow-roll" of conventional inflationary theories.

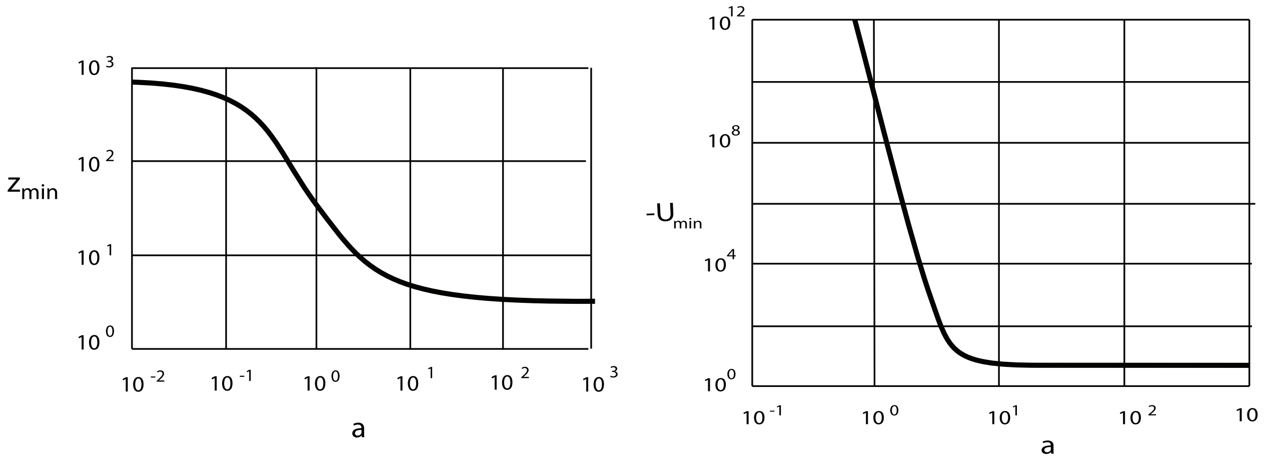

FIG. 3. The superposition of two eigenpotentials with a ratio of 50 in relative strength produces a potential with two symmetrically placed minimum that approach delta functions in the limit $a \to 0$. The scalar field becomes trapped in these minima, and the field theory approaches a spin Ising model. Here, the location of the minima $\pm z_{\rm min}$ and potential depth $U_{\rm min}$ are plotted as functions of a.

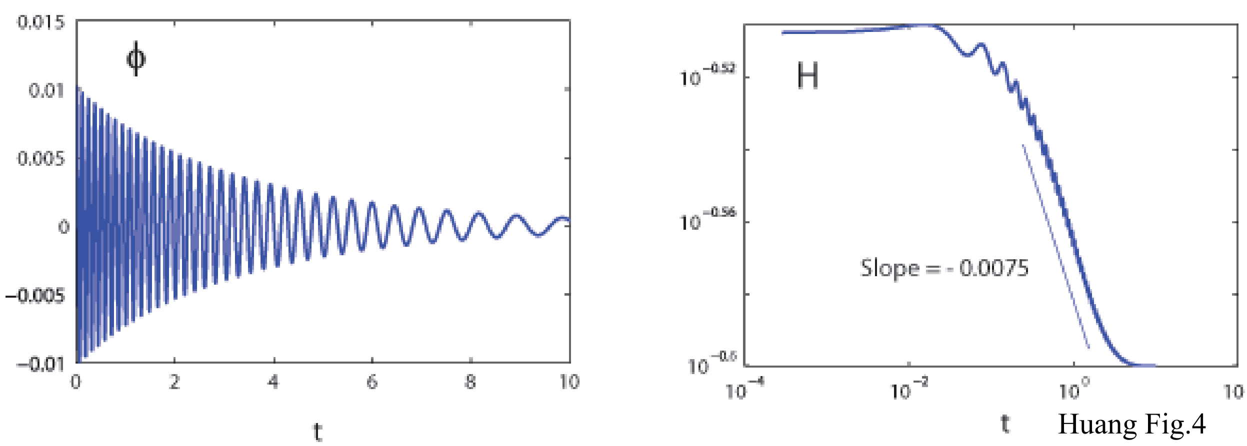

FIG. 4. Results from solving the initial-value problem with superposition of eigenpotentials depicted in Fig.3.

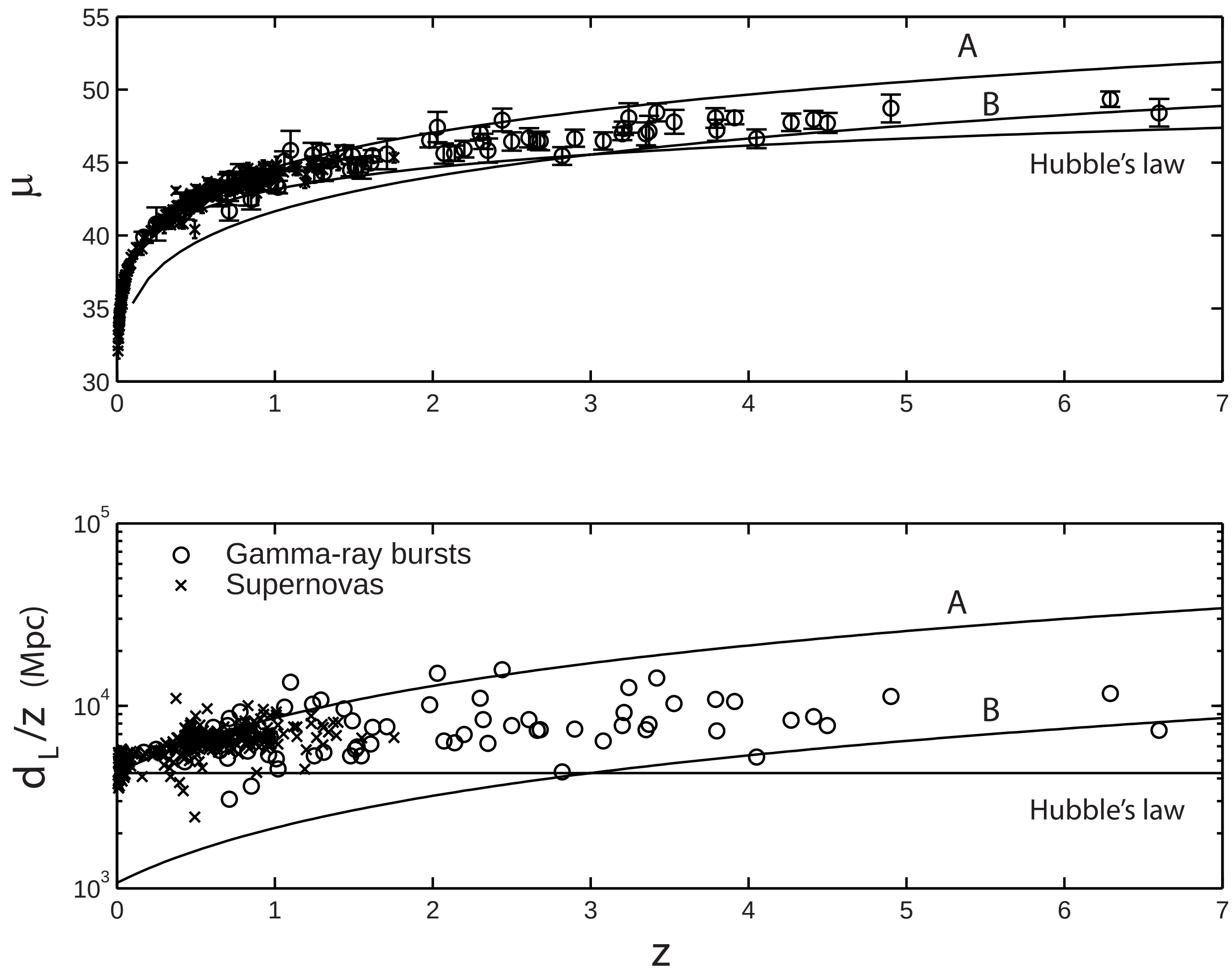

FIG. 5. Comparison between model prediction of the galactic redshift with observational data. Upper panel shows a conventional plot. Lower panel show a log-log plot of d_L/z vs z, where d_L is the luminosity distance and z is the redshift parameter. The two theoretical curves, labeled A and B, correspond to different values of the exponent p, which depends on parameters in the scalar potential, and initial conditions. See text for fuller explanation.